

# The most effective model for describing the universal behavior of unstable surface growth

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We study a noisy Kuramoto-Sivashinsky (KS) equation which describes unstable surface growth and chemical turbulence. It has been conjectured that the universal long-wavelength behavior of the equation, which is characterized by scale-dependent parameters, is described by a Kardar-Parisi-Zhang (KPZ) equation. We consider this conjecture by analyzing a renormalization-group equation for a class of generalized KPZ equations. We then uniquely determine the parameter values of the KPZ equation that most effectively describes the universal long-wavelength behavior of the noisy KS equation.

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*Introduction.*— Eddy viscosity in turbulence, which can explain how a vortex pattern emerges in a non-uniform turbulent flow, depends on the observed length scales [1]. As exemplified by the Richardson law [2], there are cases in which a parameter of a macroscopic description is not given as a definite value, but is rather expressed as a function of length scale. Another example of scale-dependent parameters has been observed in one- or two-dimensional fluid dynamics, where the viscosity is not uniquely defined in the hydrodynamic description [3]. Here, it seems reasonable to expect that such scale dependent parameters in a macroscopic description can be reproduced by an effective stochastic system [4–11]. In this Letter, we attempt to determine the effective stochastic system theoretically when scale-dependent parameters are observed.

As the simplest example for scale-dependent parameters, we consider the one-dimensional Kardar-Parisi-Zhang (KPZ) equation [12]. It is known that the effective surface tension  $\nu(\Lambda)$  at a scale  $2\pi/\Lambda$  for the equation is  $\nu(\Lambda) = C_\nu \Lambda^{-1/2}$  in the limit  $\Lambda \rightarrow 0$ , which is similar to the Richardson law for turbulence. Recently, the KPZ equation was rigorously derived from a stochastic many-particle model [13, 14], and the so-called KPZ class has been extensively discussed both theoretically and experimentally [15–20]. However, in general, even if we find systems that may exhibit scale-dependent parameters similar to those for the KPZ class, a method to determine the parameter values of the corresponding KPZ equation has not yet been reported.

Specifically, let us consider a noisy Kuramoto-Sivashinsky (KS) equation, which exhibits spatially extended chaos in the noiseless limit [21–23]. The model describes turbulent chemical waves and unstable interface motion, which are caused by negative surface tension. It has been conjectured that a KPZ equation may be an effective model for describing the long-wavelength behavior of the noisy KS equation; this conjecture is referred to as the *Yakhot conjecture* [24, 25]. Indeed, direct numerical simulations showed that statistical properties

of the long wave length modes are similar to those of the KPZ equations [26–30]. Here, one may recall the renormalization group (RG), which is a standard method for studying scale-dependent parameters. For a given noisy KS equation, the RG equation was calculated using a perturbation theory [30, 31]. The infrared fixed point of the RG equation determines the scale-dependent behavior  $\nu(\Lambda) = C_\nu \Lambda^{-1/2}$  in the limit  $\Lambda \rightarrow 0$ , which has the same power-law form as that for the KPZ equations. Nevertheless, as shown below, the analysis at the infrared fixed point of the RG equation cannot determine the parameter values of the corresponding KPZ equation.

In this Letter, we present a framework for studying the effective description. We study an RG equation for generalized KPZ equations that include noisy KS equations and KPZ equations. We then consider solution trajectories of the RG equation, in which each point flows to the infrared fixed point of the noisy KS equation we study. The solution trajectories also approach a subspace in the ultraviolet limit, which enables us to define a collection of bare parameters of the generalized KPZ equations. By using the lowest perturbation theory for the RG equation, we uniquely determine the most effective model among such KPZ equations that describes the infrared universal behavior of a noisy KS equation in the most efficient manner.

*Setup.*— We study models for the stochastic growth of a surface. We assume that the time evolution of the height  $h(x, t)$  of the surface is described by a generalized KPZ equation:

$$\partial_t h = \nu \partial_x^2 h - K \partial_x^4 h + \frac{\lambda}{2} (\partial_x h)^2 + \eta, \quad (1)$$

where  $\nu$  is the surface tension,  $K$  is the surface diffusion constant,  $\lambda$  is the strength of the non-linearity, and  $\eta(x, t)$  is the noise satisfying  $\langle \eta(x, t) \eta(x', t') \rangle = 2(D - D_d \partial_x^2) \delta(t - t') \delta(x - x')$ . Here,  $D$  and  $D_d$  are the strength of the noise. When  $K = D_d = 0$ , Eq. (1) is the KPZ equation, while when  $D = D_d = 0$ , Eq. (1) with  $\nu < 0$  is the deterministic KS equation. We refer to Eq. (1) with

$\nu < 0$  and  $K, D, D_d > 0$ , as the noisy KS equation.

The five parameters in Eq. (1) are collectively denoted by  $\mathcal{X} \equiv (\nu, K, D, D_d, \lambda)$ . More precisely, these parameters are defined for a field  $h(x)$  whose Fourier transform  $\hat{h}(k)$  is assumed to be zero for  $|k| > \Lambda$ .  $\Lambda$  is called a cut-off wavenumber. We explicitly express the cutoff dependence of the parameters as  $\mathcal{X}(\Lambda)$ . Here, for a given model with  $\mathcal{X}(\Lambda_0)$ , we define a model with  $\mathcal{X}(\Lambda)$  for  $\Lambda < \Lambda_0$  by eliminating the contribution  $\Lambda \leq |k| \leq \Lambda_0$  in the dynamics, which may be formally expressed as  $\mathcal{X}(\Lambda; \mathcal{X}(\Lambda_0))$  [32]. This functional relation trivially satisfies

$$\mathcal{X}(\Lambda'; \mathcal{X}(\Lambda_0)) = \mathcal{X}(\Lambda'; \mathcal{X}(\Lambda; \mathcal{X}(\Lambda_0))). \quad (2)$$

From this, we obtain the RG equation

$$-\Lambda \frac{d\mathcal{X}}{d\Lambda} = \Psi_{\mathcal{X}}(\mathcal{X}), \quad (3)$$

which determines  $\mathcal{X}(\Lambda; \mathcal{X}(\Lambda_0))$  under the initial condition  $\mathcal{X}(\Lambda_0) = \mathcal{X}_0$  [33].

An approximate form of  $\Psi_{\mathcal{X}}(\mathcal{X})$  with the perturbation theory [34–37] was derived in [30]. By introducing dimensionless parameters  $F \equiv \nu(\Lambda)/(K(\Lambda)\Lambda^2)$ ,  $G \equiv \lambda^2(\Lambda)D(\Lambda)/(4\pi K^3(\Lambda)\Lambda^7)$ , and  $H \equiv \lambda^2(\Lambda)D_d(\Lambda)/(4\pi K^3(\Lambda)\Lambda^5)$ , we express the result as  $\Psi_{\nu} = \nu f^3(a_0 + a_1 g)G/F$ ,  $\Psi_D = D f^3 G(1 + g)^2$ ,  $\Psi_K = K f^5(b_0 + b_1 g)G/2$ ,  $\Psi_{D_d} = D_d f^5 G^2(c_0 + c_1 g + c_2 g^2)/(2H)$ , and  $\Psi_{\lambda} = 0$ , where  $a_0 = 3 + F$ ,  $a_1 = 1 - F$ ,  $b_0 = 26 - F + 2F^2 + F^3$ ,  $b_1 = 2 - 21F + 6F^2 + F^3$ ,  $c_0 = 16 + 3F + F^2$ ,  $c_1 = 2(9 - 5F)$ ,  $c_2 = 2 - 13F - F^2$ ,  $f = 1/(1 + F)$ , and  $g = H/G$ . The result  $\Psi_{\lambda} = 0$  implies that the parameter  $\lambda$  is not renormalized, which is a consequence of the invariance of Eq. (1) under the tilt transformation [38–40].

Here, from Eq. (3), we derive the autonomous equation for  $(F(\Lambda), G(\Lambda), H(\Lambda))$ . The fixed point of the equation is calculated as  $(F^*, G^*, H^*) = (10.7593, 680.652, 63.2614)$  [41]. By using these values, we also obtain  $\nu(\Lambda) = C_{\nu}\Lambda^{-0.5}$ ,  $D(\Lambda) = C_D\Lambda^{-0.5}$ ,  $K(\Lambda) = C_K\Lambda^{-2.5}$ , and  $D_d(\Lambda) = C_{D_d}\Lambda^{-2.5}$  in the limit  $\Lambda \rightarrow 0$ , where  $C_{\nu}$ ,  $C_D$ ,  $C_K$ , and  $C_{D_d}$  are constants that depend on the initial condition  $\mathcal{X}_0$ , while

$$\frac{C_K}{C_{\nu}} = \frac{C_{D_d}}{C_D} = 0.0929 \quad (4)$$

is independent of  $\mathcal{X}_0$  [41]. The singular behavior  $\nu(\Lambda) = C_{\nu}\Lambda^{-1/2}$  implies that the effective surface tension depends on the observed scale  $\Lambda$ . This is contrasted with cases in which each  $\mathcal{X}(\Lambda)$  converges to a finite value in the limit  $\Lambda \rightarrow 0$ . Then,  $\mathcal{X}(\Lambda = 0)$  is interpreted as renormalized parameters measured in experiments. Since the exponents characterizing the divergent behavior are common to all the models given by Eq. (1), we refer to the power-law region as the universal range. The smallest characteristic wavenumber scale is also denoted by  $\Lambda_{IR}$ ,

the value of which depends on  $\mathcal{X}_0$ . Then, the universal range is defined as  $\Lambda \ll \Lambda_{IR}$ . As another common aspect of the RG equation (3), we observe that  $\mathcal{X}(\Lambda)$  shows a plateau region in the ultraviolet limit when  $\Lambda_0$  is sufficiently large. This enables us to define a collection of bare parameters, which is denoted by  $\mathcal{X}_B$ .

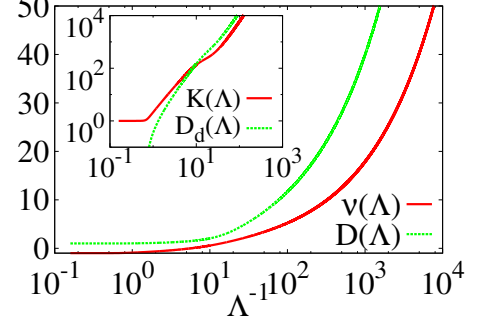


FIG. 1. (Color online) Solid (red) and dotted (green) lines show the graphs of  $\nu(\Lambda)$  and  $D(\Lambda)$  for  $\mathcal{X}_0^{KS}$ , respectively. The inset shows the graphs of  $K(\Lambda)$  (solid red) and  $D_d(\Lambda)$  (dotted green).

Here, we focus on a specific model, a noisy KS equation with  $\mathcal{X}_0^{KS}$ , ( $\nu_0 = -1.0$ ,  $D_0 = 0$ ,  $K_0 = D_{d0} = \lambda_0 = 1.0$ ), defined at  $\Lambda_0 = 2\pi$ . In Fig. 1, we display the numerical solution of Eq. (3) for this initial condition  $\mathcal{X}_0^{KS}$ . It can be seen that  $\Lambda_0$  is in the plateau region. Thus, the collection of the bare parameters  $\mathcal{X}_B^{KS}$  is assumed to be identical to the initial condition  $\mathcal{X}_0^{KS}$  without loss of accuracy [42]. On the other hand, the numerical solution in the infrared limit obeys  $\nu(\Lambda) = C_{\nu}\Lambda^{-0.5}$  and  $D(\Lambda) = C_D\Lambda^{-0.5}$  in accordance with the analysis of the fixed point.

Now, for the noisy KS equation with  $\mathcal{X}_B^{KS}$ , we consider the set  $\mathcal{B}(\mathcal{X}_B^{KS})$  of bare parameters  $\mathcal{X}_B$ , each of which has the same factors  $C_{\nu}$ ,  $C_D$ ,  $C_K$ , and  $C_{D_d}$  in the universal range and the same wavenumber scale  $\Lambda_{IR}$  as those for the noisy KS equation. The graph of  $\mathcal{X}(\Lambda)$  for a given  $\mathcal{X}_B \in \mathcal{B}(\mathcal{X}_B^{KS})$  determines the wavenumber scale  $\Lambda_{UV}$  that represents the end of the ultraviolet plateau. Note that the value of  $\Lambda_{UV}$  depends on  $\mathcal{X}_B \in \mathcal{B}(\mathcal{X}_B^{KS})$ . Then, there is a special model with  $\mathcal{X}_B \in \mathcal{B}(\mathcal{X}_B^{KS})$  such that  $\Lambda_{UV} = \Lambda_{IR}^{KS}$ . For this model, as soon as the graph of  $\mathcal{X}(\Lambda)$  exits from the ultraviolet plateau region, it enters the infrared universal range. In other words, this special model represents the universal behavior of the noisy KS equation in the most efficient manner. We refer to it as *the most effective model for the universal range of the noisy KS equation with  $\mathcal{X}_B^{KS}$* . Below, we determine the most effective model.

*Representation of the parameter space.*— Solution trajectories for the RG equation are expressed as curves in the five-dimensional parameter space consisting of  $\mathcal{X}$ . We attempt to simplify the representation of trajectories so as to determine the most effective model. First, recalling

$\Psi_\lambda = 0$ , we may restrict the parameter space into the subspace  $\lambda = \lambda_0 = 1$ .

Next, as shown in Fig. 2, we find that  $D(\Lambda)/\nu(\Lambda)$  and  $D_d(\Lambda)/K(\Lambda)$  converge to the same value, 2.24, in the universal range for the noisy KS equation. We can explain this phenomenon as follows. First, for the generalized KPZ equations with  $\mathcal{X}_B$  satisfying  $D_B/\nu_B = D_{dB}/K_B \equiv \chi > 0$ , we can show the fluctuation-dissipation relation with the effective temperature  $\chi$  fixed by using a time-reversal symmetry. This relation leads to the invariance property of  $D(\Lambda)/\nu(\Lambda)$  and  $D_d(\Lambda)/K(\Lambda)$  along solution trajectories of the RG equation (3) [43]. For the other cases where  $D_B/\nu_B \neq D_{dB}/K_B$  including for noisy KS equations,  $D(\Lambda)/\nu(\Lambda)$  and  $D_d(\Lambda)/K(\Lambda)$  change in  $\Lambda$ . However, they satisfy  $D(\Lambda)/\nu(\Lambda) = D_d(\Lambda)/K(\Lambda)$  in the universal range. Therefore, it is reasonable to conjecture that the time-reversal symmetry emerges in the universal range. Now, since the most effective model represents the universal behavior most efficiently, this special model should be in the subspace satisfying  $D_B/\nu_B = D_{dB}/K_B \equiv \chi = 2.24$ . On the basis of the results, we express the bare-parameter space by  $(\nu_B, K_B, D_B = 2.24\nu_B, D_{dB} = 2.24K_B, \lambda_B = 1)$ . For each value of  $(\nu_B, K_B)$ , we have a model that exhibits the infrared universal behavior of  $\mathcal{X}_B^{KS}$ .

Finally, for a generalized KPZ equation with  $\mathcal{X}_B$  at  $\Lambda_0$  in the ultraviolet plateau region, we consider the following scale transformation:  $X = b_x x$ ,  $T = b_t t$ , and  $H(X, T) = b_h h(x, t)$ , which yields another generalized KPZ equation with a different collection of bare parameters  $\mathcal{X}'_B$  at  $\Lambda'_0 = b_x^{-1}\Lambda_0$  in the ultraviolet plateau region. By imposing  $\chi' = \chi$  and  $\lambda' = \lambda$ , we obtain  $b_h = b_x^{1/2}$  and  $b_t = b_x^{3/2}$ . We then find that  $J \equiv K_B/\nu_B^5$  is invariant under the transformation [44]. Thus, we parameterize  $(\nu_B, K_B)$  as  $(b_x^{1/2}, b_x^{5/2}J)$ . The next problem is to determine the values of  $b_x$  and  $J$  of the most effective model for the universal range of the noisy KS equation.

*Most effective model.*— Since  $J$  is invariant under the scale transformation, the determination of  $J$  can be sep-

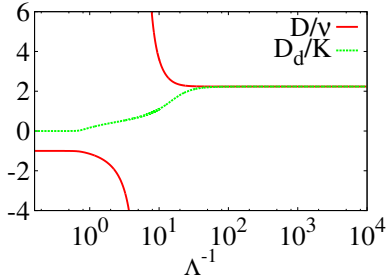


FIG. 2. (Color online) Solid (red) and dotted (green) lines show the graphs of  $D(\Lambda)/\nu(\Lambda)$  and  $D_d(\Lambda)/K(\Lambda)$  for  $\mathcal{X}_B^{KS}$ , respectively.  $D(\Lambda)/\nu(\Lambda)$  and  $D_d(\Lambda)/K(\Lambda)$  converge to the same value, 2.24.

arated from the determination of  $b_x$ . Here, we notice the condition  $\Lambda_{UV} = \Lambda_{IR}$  for the most effective model. Because this condition is invariant under the scale transformation, the value of  $J$  is uniquely determined. Furthermore, the condition  $\Lambda_{IR} = \Lambda_{IR}^{KS}$  fixes the value of  $b_x$ . Below, we explicitly calculate these values.

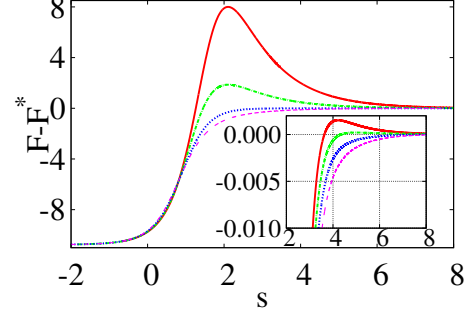


FIG. 3. (Color online) Graphs of  $F - F^*$  as a function of  $s \equiv -\ln(J^{1/2}b_x\Lambda)$  for  $J = 0.1$  (solid red line),  $J = 1.0$  (green dash-dotted line),  $J = 7.3$  (dotted blue line) and  $J = 20$  (dashed pink line). The inset shows the graphs for  $J = 7.1$  (solid red line),  $J = 7.2$  (green dash-dotted line),  $J = 7.3$  (dotted blue line), and  $J = 7.4$  (dashed pink line).

In order to determine the value of  $J$ , we study the dimensionless quantity  $F(\Lambda) = \nu(\Lambda)/(K(\Lambda)\Lambda^2)$  as a function of  $s(\Lambda) \equiv -\ln(J^{1/2}b_x\Lambda)$ , where  $F$  and  $s$  are invariant under the scale transformation. It should be noted that, for any  $J$  and  $b_x$ ,  $F$  approaches  $e^{2s}$  in the ultraviolet limit  $s \rightarrow -\infty$  and  $F$  approaches the value  $F^* = 10.76$  in the infrared limit  $s \rightarrow \infty$ . In Fig. 3, we show graphs of  $F$  as a function of  $s$  for several values of  $J$ . In general, there are two characteristic scales of  $s$ , the departure scale from  $e^{2s}$  and the relaxation scale to  $F^*$ , as clearly observed for  $J = 0.1$ . When  $J$  increases, the peak of  $F$  decreases and eventually vanishes at  $J = 4.2$ . In this case, the transition scale between the infrared universal region and the ultraviolet region is simply given by the cross point  $s_c$  of the ultraviolet behavior  $F = e^{2s}$  and the infrared behavior  $F^* = 10.8$ . That is,  $2s_c = \ln F^*$ , which gives  $s_c = 1.2$ . Thus, we conclude that the value of  $J$  of the most effective model is  $J = 4.2$ .

Next, we determine the value of  $b_x$ . From the cross point  $s_c$ , we define the transition length scale  $\Lambda_c^{-1}$  by  $s_c = -\ln(J^{1/2}b_x\Lambda_c)$ , which gives  $\Lambda_c^{-1} = \sqrt{JF^*}b_x = 8.9b_x$ . Here, the value of  $b_x$  is determined by identifying  $\Lambda_c$  with  $\Lambda_{IR}^{KS}$ . Thus, we estimate  $\Lambda_{IR}^{KS}$  from the graph of  $\nu(\Lambda)$  for the noisy KS equation under study. In Fig. 4, we show how  $\nu(\Lambda)$  approaches  $C_\nu\Lambda^{-0.5}$ . We find that  $|\nu(\Lambda) - C_\nu\Lambda^{-0.5}|$  is well fitted to a power-law function of  $\Lambda^{-1}$ , which does not provide any wavenumber scale. Through more detailed analysis, we find a fitting function

$$\nu(\Lambda) - C_\nu\Lambda^{-0.5} = -A\Lambda^B + C \exp\left[-\frac{\Lambda^{-1}}{D}\right] \quad (5)$$

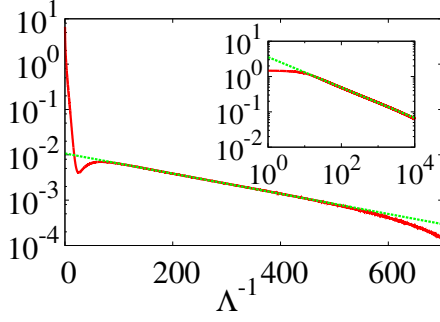


FIG. 4. (Color online) Solid (red) and dotted (green) lines show the graphs of  $|\nu(\Lambda) - C\nu\Lambda^{-1/2} + A\Lambda^B|$  for  $\mathcal{X}_B^{KS}$  and the fitted curve, respectively. The inset shows the graphs of  $|\nu(\Lambda) - C\nu\Lambda^{-1/2}|$  and  $A\Lambda^B$  with  $A = 3.57$  and  $B = 0.431$ .

with  $A = 3.57$ ,  $B = 0.431$ ,  $C = 1.1 \times 10^{-2}$ , and  $D = 195$ . From the second term of Eq. (5), we obtain the characteristic scale  $(\Lambda_{IR}^{KS})^{-1} = D = 195$ . Now, from the condition  $(\Lambda_{IR}^{KS})^{-1} = \Lambda_c^{-1} = \sqrt{JF^*}b_x = 8.9b_x$ , we obtain  $b_x = 22$ . Thus, we have arrived at the most effective model for the universal range of the noisy KS equation with  $\mathcal{X}_B^{KS}$ , where the collection of bare parameter values of the most effective model,  $\mathcal{X}_B^{ME}$ , is determined as  $(\nu_B = 4.7, D_B = 10, K_B = 1.6 \times 10^4, D_{dB} = 3.7 \times 10^4, \lambda_B = 1)$ .

Now, the linear decay rate of the disturbance of wavenumber  $k$  in the universal range is expressed as  $\nu_B k^2 + K_B k^4$  at an early time. Here, we notice that  $(\nu_B/K_B)^{0.5}$  defines one wavenumber scale. Since the most effective model has only one wavelength scale  $\Lambda_c$ ,  $(\nu_B/K_B)^{0.5} \simeq \Lambda_c$  holds. This implies that the linear decay rate  $\nu_B k^2 + K_B k^4$  is estimated as  $\nu_B k^2$  for  $k \ll \Lambda_c$ . In this manner,  $\nu_B$  can be measured in experiments. Indeed, by applying this method to the numerical simulation of the noisy KS equation, the result  $\nu_B^{\text{exp}} \simeq 5.5$  was obtained [30]. Thus, our theoretical value  $\nu_B = 4.7$  is in good agreement with the numerical value.

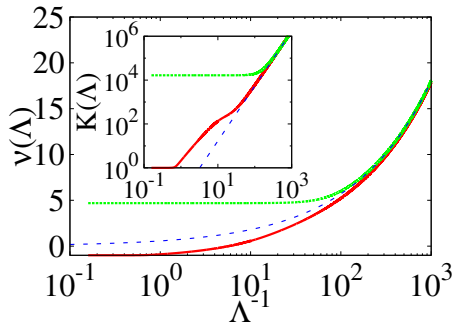


FIG. 5. (Color online) Graphs of  $\nu(\Lambda)$  and  $K(\Lambda)$  for the most effective model and the noisy KS equation, respectively. The solid (red), dotted (green), and dashed (blue) lines represent  $\nu(\Lambda)$  for  $\mathcal{X}_B^{KS}$ ,  $\mathcal{X}_B^{ME}$ , and the infrared scaling behavior, respectively. The inset shows  $K(\Lambda)$  for these cases.

*Concluding remarks.*— The main result of this Letter is illustrated in Fig. 5. For a given noisy KS equation, we construct the most effective model exhibiting the same infrared universal behavior with just one cross-over wavenumber scale  $\Lambda_{IR}^{KS}$  connecting the infrared behavior and the ultraviolet behavior. We emphasize that our theory enables us to calculate the bare viscosity  $\nu_B$  of the effective model in the universal range, which could not be obtained by previous studies. We conclude this Letter by presenting a few remarks.

The first remark is on the relevant parameter space in the universal range. Since  $\lambda$  is a conserved quantity along the solution of the RG equation, it obviously depends on the initial condition  $\mathcal{X}_0$ . Thus, it is relevant in the universal range. Furthermore,  $D/\nu - D_d/K$  is not relevant because  $D(\Lambda)/\nu(\Lambda) - D_d(\Lambda)/K(\Lambda)$  approaches zero. At the same time,  $\chi = D/\nu$  is a relevant parameter because its value is invariant along the solution trajectory when  $D_0/\nu_0 = D_{d0}/K_0$ . Finally, in the limit  $\Lambda \rightarrow 0$ ,  $K(\Lambda)\Lambda^2/\nu(\Lambda)$  approaches the universal constant value 0.0929 which is independent of  $\mathcal{X}_0$ . Thus, we can state that  $K(\Lambda)\Lambda^2/\nu(\Lambda)$  is irrelevant, following the argument in [45, 46]. In other words,  $\nu(\Lambda)$  and  $K(\Lambda)$  are not independent of each other in the universal range. In summary, the relevant parameter space in the universal range is spanned by the three parameters  $(\nu, \chi, \lambda)$ . However, the parameter  $K$  cannot be negligible because the irrelevant parameter  $K(\Lambda)\Lambda^2/\nu(\Lambda)$  is not zero in the universal range. This is different from many standard RG analysis [47].

Second, we remark that the original Yakhot conjecture claims a statistical property of the deterministic KS equation [24]. Here, we discuss the noiseless limit  $D_0 \rightarrow 0$  for the noisy KS equation. In this case, we obtain  $\chi \rightarrow 0$  which is not consistent with observations. This implies that the lowest order contribution in loop expansions is not sufficient to yield statistical properties for the small  $D_0$  limit. In order to overcome this situation, we have to formulate a non-perturbative calculation. This is an interesting problem for future work.

Finally, we expect that the concept proposed in this Letter will be applied to various systems, although we have studied a specific phenomenon as an example of scale-dependent parameters. The most interesting example may be fluid turbulence. The effective model for the universal range in turbulence is given by a noisy Navier-Stokes equation, the noise intensity of which exhibits a divergence in the infrared limit, as suggested in [5–11]. The analysis of solution trajectories of the RG equation for such a noisy Navier-Stokes equation may provide fresh insight into the understanding of turbulence. We hope that this Letter stimulates the study of whole solutions of RG equations in various research fields.

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- [1] U. Frisch, *Turbulence*, (Cambridge University Press, Cambridge, 1995).
- [2] L. F. Richardson, Proc. R. Soc. Lond. A. **110**, 709-737 (1926).
- [3] Y. Pomeau and P. Résibois, Phys. Rep. **19C** 64 (1975).
- [4] D. Forster, D. R. Nelson, and M. J. Stephen Phys. Rev. A **16**, 732 (1977).
- [5] C. DeDominicis and P. C. Martin, Phys. Rev. A. **19**, 419-421 (1979).
- [6] J. D. Fournier and U. Frisch, Phys. Rev. A **17**, 747 (1978).
- [7] J. D. Fournier and U. Frisch, Phys. Rev. A **28**, 1000 (1983).
- [8] V. Yakhot and S. A. Orszag, Phys. Rev. Lett. **57**, 1722 (1986).
- [9] V. Yakhot and S. Orszag, J. Sci. Comput. **1**, 3 (1986).
- [10] V. Yakhot and M. L. Smith, J. Sci. Comput. **7**, 35 (1992).
- [11] G. L. Eyink, Phys. Fluids **6** 3063 (1994).
- [12] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
- [13] L. Bertini and G. Giacomin, Comm. Math. Phys. **183**, 571 (1997).
- [14] T. Sasamoto and H. Spohn, Phys. Rev. Lett. **104**, 230602 (2010).
- [15] K. A. Takeuchi and M. Sano, Phys. Rev. Lett. **104**, 230601 (2010).
- [16] K. A. Takeuchi, M. Sano, T. Sasamoto and H. Spohn, Sci. Rep. **1**, 34 (2011).
- [17] G. Amir, I. Corwin, and J. Quastel, Comm. Pure Appl. Math. **64** 466 (2010).
- [18] P. Calabrese and P. LeDoussal, Phys. Rev. Lett. **106**, 250603 (2011).
- [19] T. Imamura and T. Sasamoto Phys. Rev. Lett. **108**, 190603 (2012).
- [20] M. Hairer, Annals of Mathematics **178** 559, (2013).
- [21] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. **55**, 356 (1976).
- [22] G. I. Sivashinsky, Acta Astron. **4**, 1177 (1977).
- [23] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, (Springer, Berlin, 1984).
- [24] V. Yakhot, Phys. Rev. A **24**, 642 (1981).
- [25] V. Yakhot and Z.-S. She, Phys. Rev. Lett. **60**, 1840 (1988).
- [26] S. Zaleski, Physica D **34**, 427 (1989).
- [27] K. Sneppen, J. Krug, M. H. Jensen, C. Jayaprakash, and T. Bohr, Phys. Rev. A **46**, R7351 (1992).
- [28] F. Hayot, C. Jayaprakash, and C. Josserand, Phys. Rev. E **47**, 911 (1993).
- [29] H. Sakaguchi, Prog. Theor. Phys. **107**, 879 (2002).
- [30] K. Ueno, H. Sakaguchi, and M. Okamura, Phys. Rev. E **71**, 046138 (2005).
- [31] R. Cuerno and K. B. Lauritsen, Phys. Rev. E **52**, 4853 (1995).
- [32] See Supplemental Material for the determination of  $\mathcal{X}(\Lambda; \mathcal{X}(\Lambda_0))$ .
- [33] We do not employ the rescaling transformation after the coarse-graining.
- [34] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
- [35] H. K. Janssen, Z. Phys. B **23**, 377 (1976).
- [36] C. De Dominicis, J. Phys. Colloq. **37**, C1-247 (1976).
- [37] C. De Dominicis, Phys. Rev. B **18**, 4913 (1978).
- [38] See Supplemental Material for the derivation of  $\Psi_\lambda = 0$ .
- [39] E. Frey and U. C. Täuber, Phys. Rev. E **50**, 1024 (1994).
- [40] L. Canet, H. Chaté, B. Delamotte, and N. Wschebor, Phys. Rev. E **84**, 061128 (2011).
- [41] See Supplemental Material for the derivation of  $(F^*, G^*, H^*)$ .
- [42]  $D_d(\Lambda)$  does not show the plateau region in Fig. 1, but this graph quickly converges to  $D_d(\Lambda)$  with  $D_{d0} = 0$  at  $\Lambda_0 = \infty$ . Thus, the bare parameter is defined as  $D_{dB} = 0$ . See Supplemental Material for the detail.
- [43] See Supplemental Material for the time-reversal symmetry of the generalized KPZ equation and the derivation of the fluctuation-dissipation relation.
- [44] See Supplemental Material for the explicit calculation of the scale transformation.
- [45] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).
- [46] S. Weinberg, *The quantum theory of fields*, (Cambridge university press, Cambridge, 1995).
- [47] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).
- [48] In general, by assuming time dependence of the infinitesimal parameter for a continuous symmetry transformation, we can obtain non-trivial identities such as Eq. (S37). This technique, which has been referred to as “gauging a global symmetry”, is standard when we derive identities from a continuous global symmetry [46]. For such a case, the variation of an action under a time-gauged transformation is expressed as  $\delta S = \int dt Q(t) \partial_t \epsilon(t)$ , where  $Q(t)$  is a Noether charge of the corresponding global symmetry, and  $\epsilon(t)$  is the time-gauged infinitesimal parameter. The Noether charge of the shift symmetry is calculated as  $Q_{\text{shift}} = \int dx i \hbar(x, t)$ , which is consistent with Eq. (S44).

# SUPPLEMENTAL MATERIAL

## RENORMALIZATION GROUP EQUATION

In this section, we review the renormalization group (RG) equation for the generalized Kardar-Parisi-Zhang (KPZ) equations. We first define scale-dependent parameters  $\nu(\Lambda)$ ,  $D(\Lambda)$ ,  $K(\Lambda)$ , and  $D_d(\Lambda)$ , and then introduce a perturbation theory leading to the equation for determining them.

We start with the generating functional  $Z[J, \tilde{J}]$  by which all statistical quantities of the KPZ equations are determined. Following the Martin-Siggia-Rose-Janssen-deDominicis (MSRJD) formalism [34–37],  $Z[J, \tilde{J}]$  is expressed as

$$Z[J, \tilde{J}] = \int \mathcal{D}[h, i\tilde{h}] \exp \left[ -S[h, i\tilde{h}; \Lambda_0] + \int_{-\infty}^{\infty} d\omega \int_{-\Lambda_0}^{\Lambda_0} dk \left( J(k, \omega) h(-k, -\omega) + \tilde{J}(k, \omega) i\tilde{h}(-k, -\omega) \right) \right], \quad (\text{S1})$$

where  $i\tilde{h}$  is the auxiliary field,  $J$  and  $\tilde{J}$  are source fields, and  $S[h, i\tilde{h}; \Lambda_0]$  is the MSRJD action for the generalized KPZ equation. Throughout Supplemental Material, we use the notation  $A(k, \omega)$  for the Fourier transform of  $A(x, t)$  for any field  $A$ . The action  $S[h, i\tilde{h}; \Lambda_0]$  is explicitly written as

$$\begin{aligned} S[h, i\tilde{h}; \Lambda_0] = & \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda_0}^{\Lambda_0} \frac{dk}{2\pi} (h(-k, -\omega) i\tilde{h}(-k, -\omega)) G_0^{-1}(k, \omega) \begin{pmatrix} h(k, \omega) \\ i\tilde{h}(k, \omega) \end{pmatrix} \\ & + \frac{\lambda_0}{2} \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{(2\pi)^2} \int_{-\Lambda_0}^{\Lambda_0} \frac{dk_1 dk_2}{(2\pi)^2} k_1 k_2 i\tilde{h}(-k_1 - k_2, -\omega_1 - \omega_2) h(k_1, \omega_1) h(k_2, \omega_2), \end{aligned} \quad (\text{S2})$$

where  $G_0^{-1}$  is the inverse matrix of the bare propagator

$$G_0^{-1}(k, \omega) = \begin{pmatrix} 0 & i\omega + \nu_0 k^2 + K_0 k^4 \\ -i\omega + \nu_0 k^2 + K_0 k^4 & -2(D_0 + D_{d0} k^2) \end{pmatrix}. \quad (\text{S3})$$

Here, we consider a coarse-grained description at a cutoff  $\Lambda < \Lambda_0$ . Let us define

$$A^<(k, \omega) \equiv \theta(\Lambda - k) A(k, \omega), \quad (\text{S4})$$

$$A^>(k, \omega) \equiv \theta(k - \Lambda) A(k, \omega), \quad (\text{S5})$$

for any quantity  $A(k, \omega)$ , where  $\theta(x)$  is the Heaviside step function. The statistical quantities of  $h^<$  are described by the generating functional  $Z[J^<, \tilde{J}^<]$  with replacement of  $(J, \tilde{J})$  by  $(J^<, \tilde{J}^<)$ . We thus define the effective MSRJD action  $S[h^<, i\tilde{h}^<; \Lambda]$  by the relation

$$Z[J^<, \tilde{J}^<] = \int \mathcal{D}[h^<, i\tilde{h}^<] \exp \left[ -S[h^<, i\tilde{h}^<; \Lambda] + \int_{-\infty}^{\infty} d\omega \int_{-\Lambda}^{\Lambda} dk \left( J^<(k, \omega) h^<(-k, -\omega) + \tilde{J}^<(k, \omega) i\tilde{h}^<(-k, -\omega) \right) \right]. \quad (\text{S6})$$

We can then confirm that  $S[h^<, i\tilde{h}^<; \Lambda]$  is determined as

$$\exp \left[ -S[h^<, i\tilde{h}^<; \Lambda] \right] = \int \mathcal{D}[h^>, i\tilde{h}^>] \exp \left[ -S[h^< + h^>, i\tilde{h}^< + i\tilde{h}^>; \Lambda_0] \right]. \quad (\text{S7})$$

Then, the propagator and the three point vertex function for the effective MSRJD action at  $\Lambda$  are defined as

$$(G^{-1})_{\tilde{h}h}(k_1, \omega_1; \Lambda) \delta(\omega_1 + \omega_2) \delta(k_1 + k_2) \equiv \left. \frac{\delta^2 S[h^<, i\tilde{h}^<; \Lambda]}{\delta(i\tilde{h}(k_1, \omega_1)) \delta(h^<(k_2, \omega_2))} \right|_{h^<=0, i\tilde{h}^<=0}, \quad (\text{S8})$$

$$(G^{-1})_{\tilde{h}\tilde{h}}(k_1, \omega_1; \Lambda) \delta(\omega_1 + \omega_2) \delta(k_1 + k_2) \equiv \left. \frac{\delta^2 S[h^<, i\tilde{h}^<; \Lambda]}{\delta(i\tilde{h}^<(k_1, \omega_1)) \delta(i\tilde{h}^<(k_2, \omega_2))} \right|_{h^<=0, i\tilde{h}^<=0}, \quad (\text{S9})$$

$$\Gamma_{\tilde{h}hh}(k_1, \omega_1; k_2, \omega_2; \Lambda) \delta(\omega_1 + \omega_2 + \omega_3) \delta(k_1 + k_2 + k_3) \equiv \left. \frac{\delta^3 S[h^<, i\tilde{h}^<; \Lambda]}{\delta(i\tilde{h}^<(k_1, \omega_1)) \delta(h^<(k_2, \omega_2)) \delta(h^<(k_3, \omega_3))} \right|_{h^<=0, i\tilde{h}^<=0}. \quad (\text{S10})$$

From these quantities, we define the parameters as

$$\nu(\Lambda) \equiv \lim_{\omega, k \rightarrow 0} \frac{1}{2!} \frac{\partial^2 (G^{-1})_{\tilde{h}h}(\omega, k; \Lambda)}{\partial k^2}, \quad (\text{S11})$$

$$K(\Lambda) \equiv \lim_{\omega, k \rightarrow 0} \frac{1}{4!} \frac{\partial^4 (G^{-1})_{\tilde{h}h}(\omega, k; \Lambda)}{\partial k^4}, \quad (\text{S12})$$

$$-2D(\Lambda) \equiv \lim_{\omega, k \rightarrow 0} (G^{-1})_{\tilde{h}\tilde{h}}(\omega, k; \Lambda), \quad (\text{S13})$$

$$-2D_d(\Lambda) \equiv \lim_{\omega, k \rightarrow 0} \frac{1}{2!} \frac{\partial^2 (G^{-1})_{\tilde{h}h}(\omega, k; \Lambda)}{\partial k^2}, \quad (\text{S14})$$

$$\lambda(\Lambda) \equiv \lim_{\omega_1, \omega_2, k_1, k_2 \rightarrow 0} \frac{\partial^2 \Gamma_{\tilde{h}hh}(k_1, \omega_1; k_2, \omega_2; \Lambda)}{\partial k_1 \partial k_2}. \quad (\text{S15})$$

In the next section, we will provide a non-perturbative proof for the claim that  $\lambda(\Lambda) = \lambda_0$  on the basis of symmetry properties [39]. Below, we derive a set of equations that determines  $\nu(\Lambda)$ ,  $D(\Lambda)$ ,  $K(\Lambda)$ , and  $D_d(\Lambda)$ , respectively.

We can calculate  $(G^{-1})_{ij}(k, \omega; \Lambda)$  by using the perturbation theory in  $\lambda_0$ . At the second-order level, the propagators are calculated as

$$\begin{aligned} (G^{-1})_{\tilde{h}h}(\omega, k; \Lambda) &= (G_0^{-1})_{\tilde{h}h}(k, \omega) \\ &+ \lambda_0^2 \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{\Lambda \leq |q| \leq \Lambda_0} \frac{dq}{2\pi} \left[ kq(k-q)^2 (G_0)_{\tilde{h}h}(q, \Omega) C_0(k-q, \omega-\Omega) \right. \\ &\left. + kq^2(k-q) (G_0)_{\tilde{h}h}(k-q, \omega-\Omega) C_0(q, \Omega) \right], \end{aligned} \quad (\text{S16})$$

$$\begin{aligned} (G^{-1})_{\tilde{h}\tilde{h}}(\omega, k; \Lambda) &= (G_0^{-1})_{\tilde{h}\tilde{h}}(k, \omega) \\ &- 2\lambda_0^2 \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{\Lambda \leq |q| \leq \Lambda_0} \frac{dq}{2\pi} q^2 (k-q)^2 C_0(q, \Omega) C_0(k-q, \omega-\Omega), \end{aligned} \quad (\text{S17})$$

where  $C_0(k, \omega)$  is the bare correlation function defined by

$$C_0(k, \omega) \equiv 2(D_0 + D_{d0}k^2) |(G_0)_{\tilde{h}h}(k, \omega)|^2. \quad (\text{S18})$$

In the calculation of Eq. (S16), one should carefully note the relation [30]

$$(k-q)(G_0)_{\tilde{h}h}(q, \Omega) C_0(k-q, \omega-\Omega) \neq q(G_0)_{\tilde{h}h}(k-q, \omega-\Omega) C_0(q, \Omega). \quad (\text{S19})$$

By setting  $(\Lambda - \Lambda_0)/\Lambda_0 \ll 1$  for Eqs. (S11) - (S17), we obtain the RG equation

$$-\Lambda \frac{d\nu(\Lambda)}{d\Lambda} = \nu(\Lambda) \left[ \frac{G}{F(1+F)^3} \left( 3 + F + (1-F) \frac{H}{G} \right) \right], \quad (\text{S20})$$

$$-\Lambda \frac{dK(\Lambda)}{d\Lambda} = K(\Lambda) \left[ \frac{G}{2(1+F)^5} \left( 26 - F + 2F^2 + F^3 + (2 - 21F + 6F^2 + F^3) \frac{H}{G} \right) \right], \quad (\text{S21})$$

$$-\Lambda \frac{dD(\Lambda)}{d\Lambda} = D(\Lambda) \left[ \frac{G}{(1+F)^3} \left( 1 + \frac{H}{G} \right)^2 \right], \quad (\text{S22})$$

$$-\Lambda \frac{dD_d(\Lambda)}{d\Lambda} = D_d(\Lambda) \left[ \frac{G^2}{2H(1+F)^5} \left( 16 + 3F + F^2 + 2(9 - 5F) \frac{H}{G} + (2 - 13F - F^2) \frac{H^2}{G^2} \right) \right], \quad (\text{S23})$$

where we have introduced the dimensionless parameters  $F$ ,  $G$  and  $H$  as

$$F = \frac{\nu(\Lambda)}{K(\Lambda)\Lambda^2}, \quad (\text{S24})$$

$$G = \frac{\lambda_0^2 D(\Lambda)}{4\pi K^3(\Lambda)\Lambda^7}, \quad (\text{S25})$$

$$H = \frac{\lambda_0^2 D_d(\Lambda)}{4\pi K^3(\Lambda)\Lambda^5}. \quad (\text{S26})$$

We also obtain the following equations of  $F$ ,  $G$  and  $H$  from Eqs. (S20) - (S23):

$$-\Lambda \frac{dF}{d\Lambda} = 2F + \frac{G}{2(1+F)^5} \left[ 6 - 12F + 11F^2 - F^4 + (2 + 19F^2 - 8F^3 - F^4) \frac{H}{G} \right], \quad (\text{S27})$$

$$-\Lambda \frac{dG}{d\Lambda} = 7G - \frac{G^2}{2(1+F)^5} \left[ 76 - 7F + 4F^2 + 3F^3 + (2 - 71F + 14F^2 + 3F^3) \frac{H}{G} - 2(1+F)^2 \frac{H^2}{G^2} \right], \quad (\text{S28})$$

$$-\Lambda \frac{dH}{d\Lambda} = 5H + \frac{G^2}{2(1+F)^5} \left[ 16 + 3F + F^2 - (60 + 7F + 6F^2 + 3F^3) \frac{H}{G} - (4 - 50F + 19F^2 + 3F^3) \frac{H^2}{G^2} \right]. \quad (\text{S29})$$

The stable fixed point of the equations Eqs. (S27) - (S29) is found to be  $(F^*, G^*, H^*) = (10.7593, 680.652, 63.2614)$ . By substituting the fixed point values to Eqs. (S20)-(S23) and solving them, we obtain the scaling laws

$$\nu(\Lambda) = C_\nu \Lambda^{-0.5}, \quad (\text{S30})$$

$$D(\Lambda) = C_D \Lambda^{-0.5}, \quad (\text{S31})$$

$$K(\Lambda) = C_K \Lambda^{-2.5}, \quad (\text{S32})$$

$$D_d(\Lambda) = C_{D_d} \Lambda^{-2.5} \quad (\text{S33})$$

in the limit  $\Lambda \rightarrow 0$ . Here,  $C_\nu$ ,  $C_D$ ,  $C_K$  and  $C_{D_d}$  are constants which are determined by the initial parameter values.

### WARD-TAKAHASHI IDENTITIES

In this section, we prove

$$\lambda(\Lambda) = \lambda_0 \quad (\text{S34})$$

for all generalized KPZ equations, and

$$\frac{\nu(\Lambda)}{D(\Lambda)} = \frac{\nu_0}{D_0} \quad (\text{S35})$$

for  $K_0 = D_{d0} = 0$  or  $K_0/D_{d0} = \nu_0/D_0$ , and

$$\frac{K(\Lambda)}{D_d(\Lambda)} = \frac{\nu_0}{D_0} \quad (\text{S36})$$

for  $K_0/D_{d0} = \nu_0/D_0$ . These results are easily obtained from the following Ward-Takahashi identities [39, 40]:

$$(G^{-1})_{\bar{h}h}(k=0, \omega; \Lambda) = -i\omega, \quad (\text{S37})$$

$$i\lambda_0 k_1 \partial_{\omega_1} (G^{-1})_{\bar{h}h}(k_1, \omega_1; \Lambda) = \lim_{\omega, k \rightarrow 0} \partial_k \Gamma_{\bar{h}hh}(k_1, \omega_1; k, \omega; \Lambda), \quad (\text{S38})$$

and

$$G_{\bar{h}h}^{-1}(k_1, \omega_1; \Lambda) + G_{\bar{h}h}^{-1}(-k_1, -\omega_1; \Lambda) = -\frac{\nu_0 k_1^2}{D_0} G_{\bar{h}h}^{-1}(k_1, -\omega_1; \Lambda). \quad (\text{S39})$$

These identities are related to invariance properties of the MSRJD action for the shift transformation, tilt transformation, and the time-reversal transformation, respectively. In the next subsections, we will derive Eqs. (S37)-(S39) following the arguments [39, 40].

Here, we derive Eqs. (S34)-(S36) from Eqs. (S37)-(S39). First, by differentiating Eq. (S38) with respect to  $k_1$  and taking the limit  $k_1 \rightarrow 0$ , we have

$$i\lambda_0 \partial_{\omega_1} (G^{-1})_{\bar{h}h}(k_1=0, \omega_1; \Lambda) = \lim_{\omega, k, k_1 \rightarrow 0} \partial_k \partial_{k_1} \Gamma_{\bar{h}hh}(k_1, \omega_1; k, \omega; \Lambda). \quad (\text{S40})$$

Next, we substitute Eq. (S37) to Eq. (S40) and take the limit  $\omega_1 \rightarrow 0$ . Then, we obtain

$$\lambda_0 = \lim_{\omega, \omega_1, k, k_1 \rightarrow 0} \partial_k \partial_{k_1} \Gamma_{\bar{h}hh}(k_1, \omega_1; k, \omega; \Lambda). \quad (\text{S41})$$



By recalling the definition Eq. (S15), we find that this equality is Eq. (S34). Second, we differentiate Eq.(S39) twice with respect to  $k_1$ . Then, we have

$$\partial_{k_1}^2 G_{\tilde{h}h}^{-1}(k_1, \omega_1; \Lambda) + \partial_{k_1}^2 G_{\tilde{h}h}^{-1}(-k_1, -\omega_1; \Lambda) = -\frac{\nu_0}{D_0}(2 + 2k_1\partial_{k_1} + k_1^2\partial_{k_1}^2)G_{\tilde{h}h}^{-1}(k_1, -\omega_1; \Lambda). \quad (\text{S42})$$

By taking the limit  $\omega_1, k_1 \rightarrow 0$  and using Eqs. (S11) and (S13), we obtain Eq. (S35). Finally, by differentiating Eq. (S39) four times with respect to  $k_1$ , we arrive at Eq. (S36).

### Proof of Eq. (S37)

We consider a shift transformation

$$h'(x, t) = h(x, t) + c(t), \quad (\text{S43})$$

where  $c(t)$  is an infinitesimal parameter that depends on time. The variation of the MSRJD action for the transformation is calculated as

$$S[h', i\tilde{h}'; \Lambda_0] - S[h, i\tilde{h}; \Lambda_0] = \int dt dx i\tilde{h}(x, t) \partial_t c(t). \quad (\text{S44})$$

It should be noted that this simple form comes from the invariance property of the MSRJD action for the time-independent  $c$  [48]. Then, the variation of the effective MSRJD action is derived as

$$\begin{aligned} S[h^{<'}, i\tilde{h}^{<'}; \Lambda] &= -\log \int \mathcal{D}[h^{>'}, i\tilde{h}^{>'}] \exp \left[ -S[h', i\tilde{h}'; \Lambda_0] \right], \\ &= -\log \int \mathcal{D}[h^{>}, i\tilde{h}^{>}] \exp \left[ -S[h, i\tilde{h}; \Lambda_0] - \int dt dx i\tilde{h}(x, t) \partial_t c(t) \right], \\ &= \int dt dx i\tilde{h}^{<}(x, t) \partial_t c(t) - \log \int \mathcal{D}[h^{>}, i\tilde{h}^{>}] \exp \left[ -S[h, i\tilde{h}; \Lambda_0] - \int dt dx i\tilde{h}^{>}(x, t) \partial_t c(t) \right], \\ &= S[h^{<}, i\tilde{h}^{<}; \Lambda] + \int dt dx i\tilde{h}^{<}(x, t) \partial_t c(t). \end{aligned} \quad (\text{S45})$$

When we obtain the fourth line in Eq. (S45) from the third line, we have used

$$\begin{aligned} \int dt dx i\tilde{h}^{>}(x, t) \partial_t c(t) &= \int dt \partial_t c(t) \left( \int dx i\tilde{h}^{>}(x, t) \right), \\ &= \int dt \partial_t c(t) i\tilde{h}^{>}(k=0, t), \\ &= 0. \end{aligned} \quad (\text{S46})$$

Here, noting the trivial relation

$$S[h^{<'}, i\tilde{h}^{<'}; \Lambda] = S[h^{<}, i\tilde{h}^{<}; \Lambda] + \int dt dx \frac{\delta S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(x, t)} c(t), \quad (\text{S47})$$

we rewrite Eq. (S45) as

$$\int dt dx \left( \frac{\delta S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(x, t)} c(t) - i\tilde{h}^{<}(x, t) \partial_t c(t) \right) = 0, \quad (\text{S48})$$

which is further expressed as

$$\int dt dx \left( \frac{\delta S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(x, t)} + \partial_t i\tilde{h}^{<}(x, t) \right) c(t) = 0. \quad (\text{S49})$$

Since this equality holds for any  $c(t)$ , we obtain

$$\int dx \left( \frac{\delta S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(x, t)} + \partial_t i\tilde{h}^{<}(x, t) \right) = 0. \quad (\text{S50})$$

The differentiation of Eq. (S50) with respect to  $i\tilde{h}^<(t', x')$  leads to

$$\int dx \left( (G^{-1})_{\tilde{h}h}(x' - x, t' - t; \Lambda) + \partial_t \delta(t - t') \delta(x - x') \right) = 0. \quad (\text{S51})$$

By performing the Fourier transformation, we arrive at Eq. (S37).

### Proof of Eq. (S38)

We consider a tilt transformation

$$h'(x, t) = h(x + \lambda_0 v t, t) + v x, \quad (\text{S52})$$

$$\tilde{h}'(x, t) = \tilde{h}(x + \lambda_0 v t, t), \quad (\text{S53})$$

where  $v$  is an infinitesimal parameter. The tilt transformation for their Fourier transforms is expressed as

$$h'(k, t) = e^{i\lambda_0 v k t} h(k, t) - i v \partial_k \delta(k), \quad (\text{S54})$$

$$i\tilde{h}'(k, t) = e^{i\lambda_0 v k t} h(k, t). \quad (\text{S55})$$

We then find the symmetry property

$$S[h^{<'}, i\tilde{h}^{<'}, i\tilde{h}^{>'}; \Lambda_0] = S[h^{<} + h^{>}, i\tilde{h}^{<} + i\tilde{h}^{>}; \Lambda_0], \quad (\text{S56})$$

from which we obtain

$$\begin{aligned} S[h^{<}, i\tilde{h}^{<}; \Lambda] &= -\log \int \mathcal{D}[h^{>}, i\tilde{h}^{>}] \exp[-S[h^{<} + h^{>}, i\tilde{h}^{<} + i\tilde{h}^{>}; \Lambda_0]], \\ &= -\log \int \mathcal{J} \mathcal{D}[h^{>'}, i\tilde{h}^{>'}] \exp[-S[h^{<'}, i\tilde{h}^{<'}, i\tilde{h}^{>'}; \Lambda_0]], \\ &= S[h^{<'}, i\tilde{h}^{<'}; \Lambda] - \log \mathcal{J}, \end{aligned} \quad (\text{S57})$$

where  $\mathcal{J} = 1 + v a$  is the Jacobian for the tilt transformation, and  $a$  is a field independent quantity. The expansion of Eq. (S57) in  $v$  leads to the identity

$$\int dk dt \left[ i\lambda_0 k t \left( \frac{\delta S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(k, t)} h^{<}(k, t) + \frac{\delta S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta i\tilde{h}^{<}(k, t)} i\tilde{h}^{<}(k, t) \right) + i\delta(k) \partial_k \frac{\delta S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(k, t)} - a \right] = 0. \quad (\text{S58})$$

We differentiate this identity with respect to  $i\tilde{h}^{<}(k_1, t_1)$  and  $h^{<}(k_2, t_2)$ . Then, we have

$$\begin{aligned} \int dk dt \left[ i\lambda_0 k t \left( \frac{\delta^2 S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(k, t) \delta i\tilde{h}^{<}(k_1, t_1)} \delta(k_2 - k) \delta(t_2 - t) + \frac{\delta^3 S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(k, t) \delta i\tilde{h}^{<}(k_1, t_1) h^{<}(k_2, t_2)} h^{<}(k, t) \right. \right. \\ \left. \left. + \frac{\delta^2 S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta i\tilde{h}^{<}(k, t) h^{<}(k_2, t_2)} \delta(k_1 - k) \delta(t_1 - t) + \frac{\delta^3 S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(k, t) \delta i\tilde{h}^{<}(k_1, t_1) h^{<}(k_2, t_2)} i\tilde{h}^{<}(k, t) \right) \right. \\ \left. + i\delta(k) \partial_k \frac{\delta S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta h^{<}(k, t) \delta i\tilde{h}^{<}(k_1, t_1) h^{<}(k_2, t_2)} \right] = 0. \end{aligned} \quad (\text{S59})$$

By taking the limit  $i\tilde{h}^{<}, h^{<} \rightarrow 0$  and recalling the definitions given in Eqs. (S8) - (S10), we obtain

$$\lambda_0(k_1 t_1 + k_2 t_2) (G^{-1})_{\tilde{h}h}(k_1, t_1 - t_2; \Lambda) \delta(k_1 + k_2) = -i \lim_{k \rightarrow 0} \partial_k \int dt \Gamma_{\tilde{h}hh}(k, k_1, k_2; t - t_1, t_2 - t_1; \Lambda) \delta(k + k_1 + k_2). \quad (\text{S60})$$

The Fourier transform of this equality is Eq. (S38).

**Proof of Eq.(S39)**

We consider a time-reversal transformation

$$h'(k, \omega) = -h(k, -\omega), \quad (\text{S61})$$

$$\tilde{h}'(k, \omega) = \tilde{h}(k, -\omega) - \frac{\nu_0 k^2}{D_0} h(k, -\omega). \quad (\text{S62})$$

The variation of the action Eq. (S2) under this transformation is calculated as

$$\begin{aligned} \delta S &\equiv S[h', i\tilde{h}'; \Lambda_0] - S[h, i\tilde{h}; \Lambda_0], \\ &= \left( \frac{D_0}{\nu_0} - \frac{D_{d0}}{K_0} \right) \frac{\nu_0 K_0}{D_0} \int \frac{d\omega dk}{(2\pi)^2} \left( \frac{\nu_0}{D_0} k^2 h(-\omega, -k) h(\omega, k) - 2i\tilde{h}(-\omega, -k) h(\omega, k) \right). \end{aligned} \quad (\text{S63})$$

The generalized KPZ equation is invariant when  $D_0/\nu_0 = D_{d0}/K_0$  or  $K_0 = D_{d0} = 0$ .

Here, we focus on the case  $D_0/\nu_0 = D_{d0}/K_0$ . Then, we obtain

$$S[h^{<'}, i\tilde{h}^{<}; \Lambda] = S[h^{<}, i\tilde{h}^{<}; \Lambda] - \log \mathcal{J}, \quad (\text{S64})$$

where  $\mathcal{J}$  is the Jacobian of the time-reversal transformation. By differentiating this equality with respect to  $i\tilde{h}^{<}(k_1, \omega_1)$  and  $h^{<}(k_2, \omega_2)$ , we have

$$\begin{aligned} \frac{\delta^2 S[h^{<}, i\tilde{h}^{<}; \Lambda]}{\delta(i\tilde{h}^{<}(k_1, \omega_1))\delta(h^{<}(k_2, \omega_2))} &= -\frac{\nu_0 k_1^2}{D_0} \frac{\delta^2 S[h^{<'}, i\tilde{h}^{<}'; \Lambda]}{\delta(i\tilde{h}^{<'}(k_1, -\omega_1))\delta(i\tilde{h}^{<'}(k_2, -\omega_2))} \\ &\quad - \frac{\delta^2 S[h^{<'}, i\tilde{h}^{<}'; \Lambda]}{\delta(h^{<'}(k_1, -\omega_1))\delta(i\tilde{h}^{<'}(k_2, -\omega_2))}, \end{aligned} \quad (\text{S65})$$

where we have used the relation

$$\frac{\delta}{\delta h(k, \omega)} = -\frac{\delta}{\delta h'(k, -\omega)} - \frac{\nu_0 k^2}{D_0} \frac{\delta}{\delta i\tilde{h}'(k, -\omega)}, \quad (\text{S67})$$

$$\frac{\delta}{\delta i\tilde{h}(k, \omega)} = \frac{\delta}{\delta i\tilde{h}'(k, -\omega)}. \quad (\text{S68})$$

By recalling the definition given in Eqs. (S8) - (S10), we obtain

$$G_{\tilde{h}h}^{-1}(k_1, \omega_1; \Lambda) \delta(\omega_1 + \omega_2) \delta(k_1 + k_2) = -\left( \frac{\nu_0 k_1^2}{D_0} G_{\tilde{h}h}^{-1}(k_1, -\omega_1; \Lambda) + G_{\tilde{h}h}^{-1}(-k_1, -\omega_1; \Lambda) \right) \delta(\omega_1 + \omega_2) \delta(k_1 + k_2). \quad (\text{S69})$$

By rearranging Eq. (S69), we arrive at the identities Eq. (S39).

**DERIVATION OF EQ. (4)**

We consider the dimensionless quantities given by

$$\frac{1}{F} = \frac{K(\Lambda)\Lambda^2}{\nu(\Lambda)}, \quad (\text{S70})$$

$$\frac{H}{G} = \frac{D_d(\Lambda)\Lambda^2}{D(\Lambda)}. \quad (\text{S71})$$

Substituting the scaling relations Eqs. (S30) - (S33) to these equalities, we have

$$\begin{aligned} \frac{1}{F^*} &= \frac{C_K}{C_\nu}, \\ \frac{H^*}{G^*} &= \frac{C_{D_d}}{C_D}. \end{aligned} \quad (\text{S72})$$

On the other hand,  $(F, G, H)$  takes the value  $(10.7593, 680.652, 63.2614)$  in the limit  $\Lambda \rightarrow 0$ . Then, we arrive at

$$\frac{C_K}{C_\nu} = \frac{C_{D_d}}{C_D} = 0.0929. \quad (\text{S73})$$

This value is determined only by the fixed point.

### SCALE TRANSFORMATION

We consider the following transformation:

$$X = b_x x, \quad (\text{S74})$$

$$T = b_t t, \quad (\text{S75})$$

$$H(X, T) = b_h h(x, t), \quad (\text{S76})$$

where  $b_x$ ,  $b_t$  and  $b_h$  are constants. For the cases that  $D = \chi\nu$  and  $D_d = \chi K$ , the equation for  $H(X, T)$  is written as

$$\partial_T H = \nu' \partial_X^2 H - K' \partial_X^4 H + \frac{\lambda}{2} (\partial_X H)^2 + F, \quad (\text{S77})$$

$$\langle F(X, T) F(X', T') \rangle = 2\chi' (\nu' - K' \partial_X^2) \delta(T - T') \delta(X - X'), \quad (\text{S78})$$

where we have introduced

$$\nu' = b_t^{-1} b_x^2 \nu, \quad (\text{S79})$$

$$K' = b_t^{-1} b_x^4 K, \quad (\text{S80})$$

$$\lambda' = b_t^{-1} b_x^2 b_h^{-1} \lambda, \quad (\text{S81})$$

$$F(X, T) = b_t^{-1} b_h \eta(x, t), \quad (\text{S82})$$

$$\chi' = b_x^{-1} b_h^2 \chi. \quad (\text{S83})$$

By imposing  $\lambda' = \lambda$  and  $\chi' = \chi$ , we obtain  $b_h = b_x^{1/2}$  and  $b_t = b_x^{3/2}$ . Then, we have the relation

$$\nu' = b_x^{1/2} \nu, \quad (\text{S84})$$

$$K' = b_x^{5/2} K, \quad (\text{S85})$$

which leads to

$$\frac{K'}{\nu'^5} = \frac{K}{\nu^5}. \quad (\text{S86})$$

That is,  $J \equiv K/\nu^5$  is invariant under the scale transformation.

### BARE PARAMETER $D_{dB}$ FOR $D_{d0} = 0$

As shown in Fig. S1, the graphs of  $D_d(\Lambda)$  with  $D_{d0} = 0$  at  $\Lambda_0 = 2\pi$ ,  $10\pi$  and  $1000\pi$  do not exhibit plateau. Instead, the graphs at  $\Lambda_0 = 2\pi$  and  $10\pi$  quickly approach  $D_d(\Lambda)$  in the limit  $\Lambda_0 = \infty$  when  $\Lambda$  is smaller than  $\Lambda_0$ . Therefore, we define the bare parameter as  $D_{dB} = 0$  for such cases.

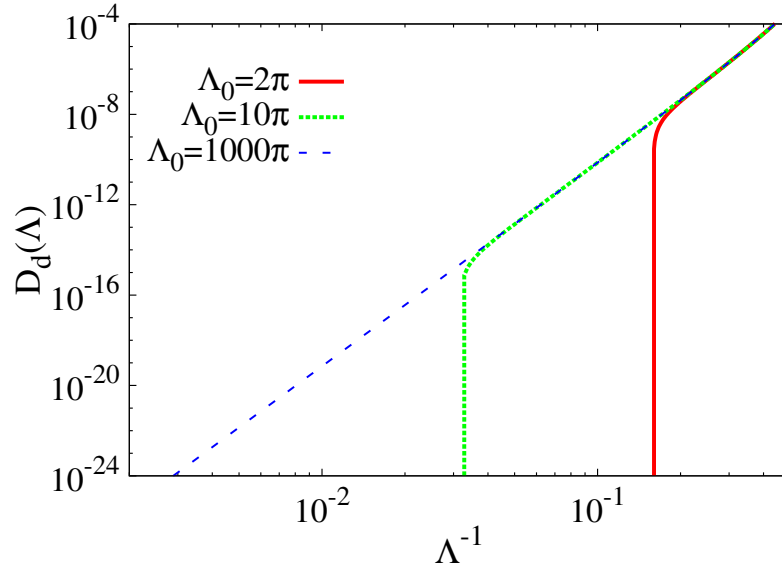


FIG. S1. (Color online) The (red) solid, (green) dotted, (blue) dashed lines show the graph of  $D_d(\Lambda)$  for  $\mathcal{X}_0^{KS}$  at  $\Lambda_0 = 2\pi$ ,  $10\pi$  and  $1000\pi$ .